Linear Classification

The Linear Model

In the next few lectures we will

- extend the perceptron learning algorithm to handle non-linearly separable data,
- explore online versis batch learning,
- learn three different learning settings – classification, regression, and probability estimation
- learn a fundamental concept in machine learning: gradient descent
- see how the learning rate hyperparameter

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The Linear Model

Recall that the linear model for binary classification is:

$$\mathcal{H} = \{h(\vec{x}) = sign(\vec{w}^T \cdot \vec{x})\}$$

where

$$\vec{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_d \end{bmatrix} \in \mathbb{R}^{d+1} \qquad \vec{x} = \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_d \end{bmatrix} \in \{1\} \times \mathbb{R}^d$$

Where

- $\vec{w} \in \mathbb{R}^{d+1}$ where d is the dimensionality of the input space and w_0 is a bias weight, and
- $x_0 = 1$ is fixed.

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Perceptron Learning Algorithm

Recall the perceptron learning algorithm, slightly reworded:

INPUT: a data set $\mathcal{D}$ with each $\vec{x_i}$ in $\mathcal{D}$ prepended with a 1, and labels $\vec{y}$

1. Initialize $\vec{w} = (w_0, w_1, ..., w_d)$ with zeros or random values, $t = 1$
2. Receive a $\vec{x_i} \in \mathcal{D}$ for which $sign(\vec{w}^T \cdot \vec{x}) \neq y_i$
   - Update $\vec{w}$ using the update rule:
     - $\vec{w}(t+1) \leftarrow \vec{w}(t) + y_i \vec{x_i}$
   - $t \leftarrow t + 1$
3. If there is still a $\vec{x_i} \in \mathcal{D}$ for which $sign(\vec{w}^T \cdot \vec{x}) \neq y_i$, repeat Step 2.

TERMINATION: $\vec{w}$ is a line separating the two classes (assuming $\mathcal{D}$ is linearly separable).

Notice that the algorithm only updates the model based on a single sample. Such an algorithm is called an *online* learning algorithm.

Also remember that PLA requires that $\mathcal{D}$ be linearly separable.

Two Fundamental Goals In Learning

We have two fundamental goals with our learning algorithms:

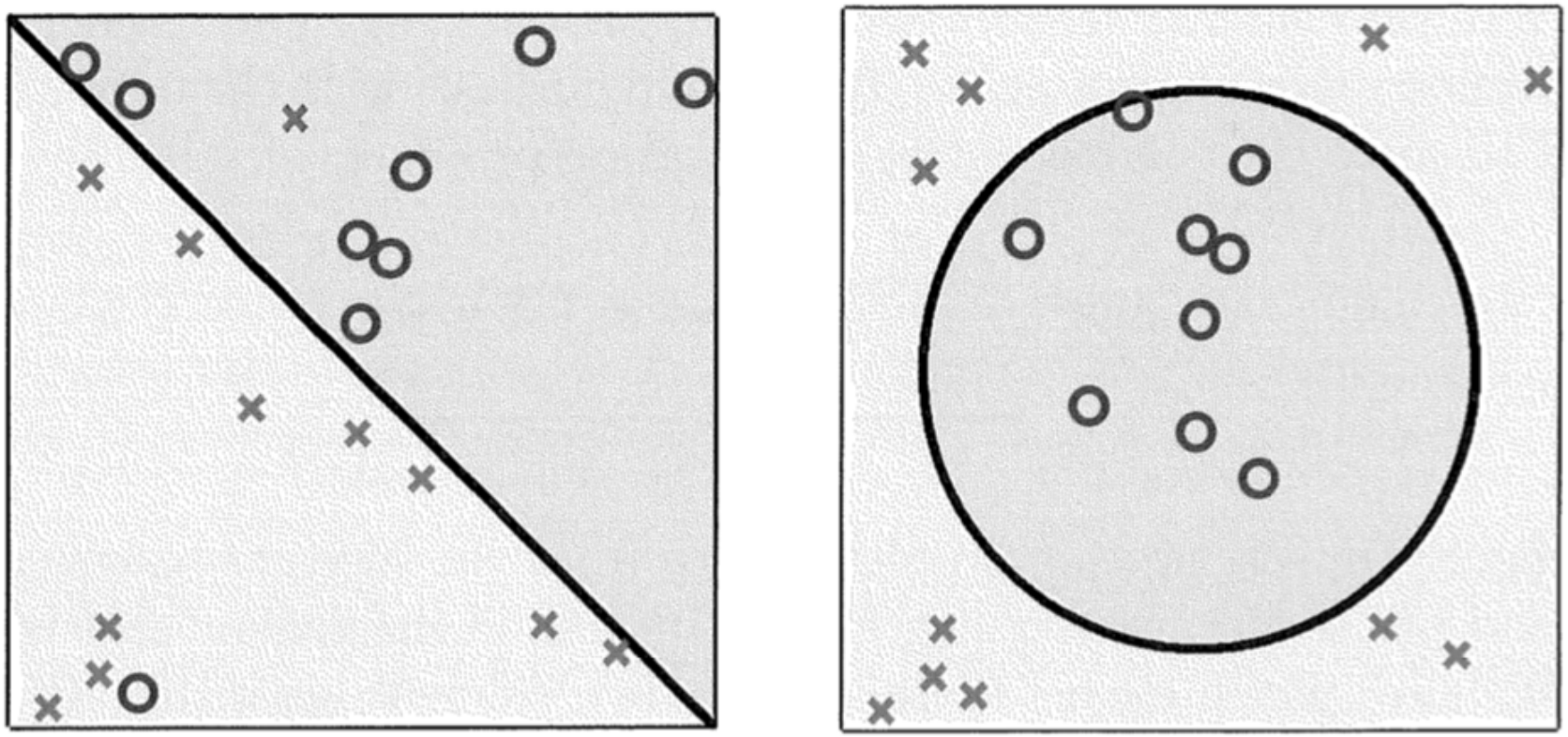
- Make $E_{out}(g)$ close to $E_{in}(g)$[1]. This means that our model will generallize well. We'll learn how to bound the difference when we study computational learning theory.
- Make $E_{in}(g)$ small. This means we have a model that fits the data well, or performs well in its prediction task.

Let's now discuss how to make $E_{in}(g)$ small. We need to define *small* and we need to deal with non-separable data.

[1] Remember that a *version space* is the set of all h in $\mathcal{H}$ consistent with our training data. g is the particular h chosen by the algorithm.

Non-Separable Data

In practice perfectly linearly separable data is rare.



(a) Few noisy data.

(b) Nonlinearly separable.

Figure 1: Figure 3.1 from Learning From Data

- Data set could include noise which prevents linear separablility.
- Data might be fundamentally non-linearly separable.

Minimizing the Error Rate

Earlier in the course we said that every machine learning problem contains the following elements:

- An input $\vec{x}$
- An unkown target function $f : \mathcal{X} \to \mathcal{Y}$
- A data set $\mathcal{D}$
- A learning model, which consists of
  - a hypothesis class $\mathcal{H}$ from which our model comes,
  - a loss function that quantifies the badness of our model, and
  - a learning algorithm which optimizes the loss function.

Error, E, is another term for loss function. For the case of our simple perceptron classifer we're using 0-1 loss, that is, counting the errors (or proportion thereof) and our optmization procedure tries to find:

$$\min_{\vec{w} \in \mathbb{R}^{d+1}} \frac{1}{N} \sum_{n=1}^{N} [\![sign(\vec{w}^T \vec{x_n}) \neq y_n]\!]$$

Let's look at two modifications to the PLA that perform this

Batch PLA [2]

INPUT: a data set $\mathcal{D}$ with each $\vec{x_i}$ in $\mathcal{D}$ prepended with a 1, labels $\vec{y}$, ϵ – an error tolerance, and α – a learning rate

1. Initialize $\vec{w} = (w_0, w_1, ..., w_d)$ with zeros or random values, $t = 1$, $\Delta = (0, ..0)$
2. do
   - For $i = 1, 2, ..., N$
     - if $sign(\vec{w}^T \cdot \vec{x_i}) \neq y_i$, then $\Delta \leftarrow \Delta + y_i \vec{x_i}$
     - $\Delta \leftarrow \frac{\Delta}{N}$
   - $\vec{w} \leftarrow \vec{w} + \alpha\Delta$

   while $||\Delta||_2 > \epsilon$

TERMINATION: $\vec{w}$ is "close enough" a line separating the two classes.

[2] Based on Alan Fern via Byron Boots

New Concepts in the Batch PLA Algorithm

1. Initialize $\vec{w} = (w_0, w_1, ..., w_d)$ with zeros or random values, $t = 1$, $\Delta = (0, ..0)$
2. do
   - For $i = 1, 2, ..., N$
     - if $sign(\vec{w}^T \cdot \vec{x_i}) \neq y_i$, then $\Delta \leftarrow \Delta + y_i \vec{x_i}$
     - $\Delta \leftarrow \frac{\Delta}{N}$
   - $\vec{w} \leftarrow \vec{w} + \alpha\Delta$

   while $||\Delta||_2 > \epsilon$

Notice a few new concepts in the batch PLA algorithm:

- the inner loop. This is a *batch algorithm* – it uses every sample in the data set to update the model.
- the ϵ hyperparameter – our stopping condition is "good enough", i.e., within an error tolerance
- the α (also sometimes η) hyperparameter – the learning rate, i.e., how much do we update the model in a given step.

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Pocket Algorithm

Input: a data set $\mathcal{D}$ with each $\vec{x_i}$ in $\mathcal{D}$ prepended with a 1, labels $\vec{y}$, and T steps

1. Initialize $\vec{w} = (w_0, w_1, ..., w_d)$ with zeros or random values, $t = 1$, $\Delta = (0, ..0)$
2. for $t = 1, 2, ..., T$
   - Run PLA for one update to obtain $\vec{w}(t+1)$
   - Evaluate $E_{in}(\vec{w}(t+1))$
     - if $E_{in}(\vec{w}(t+1)) < E_{in}(\vec{w}(t))$, then $\vec{w} \leftarrow \vec{w}(t+1)$
3. On termination, $\vec{w}$ is the best line found in T steps.

Notice

- there is an inner loop under Step 2 to evaluate E_{in}. This is also a *batch algorithm* – it uses every sample in the data set to update the model.
- the T hyperparameter simply sets a hard limit on the number of learning iterations we perform

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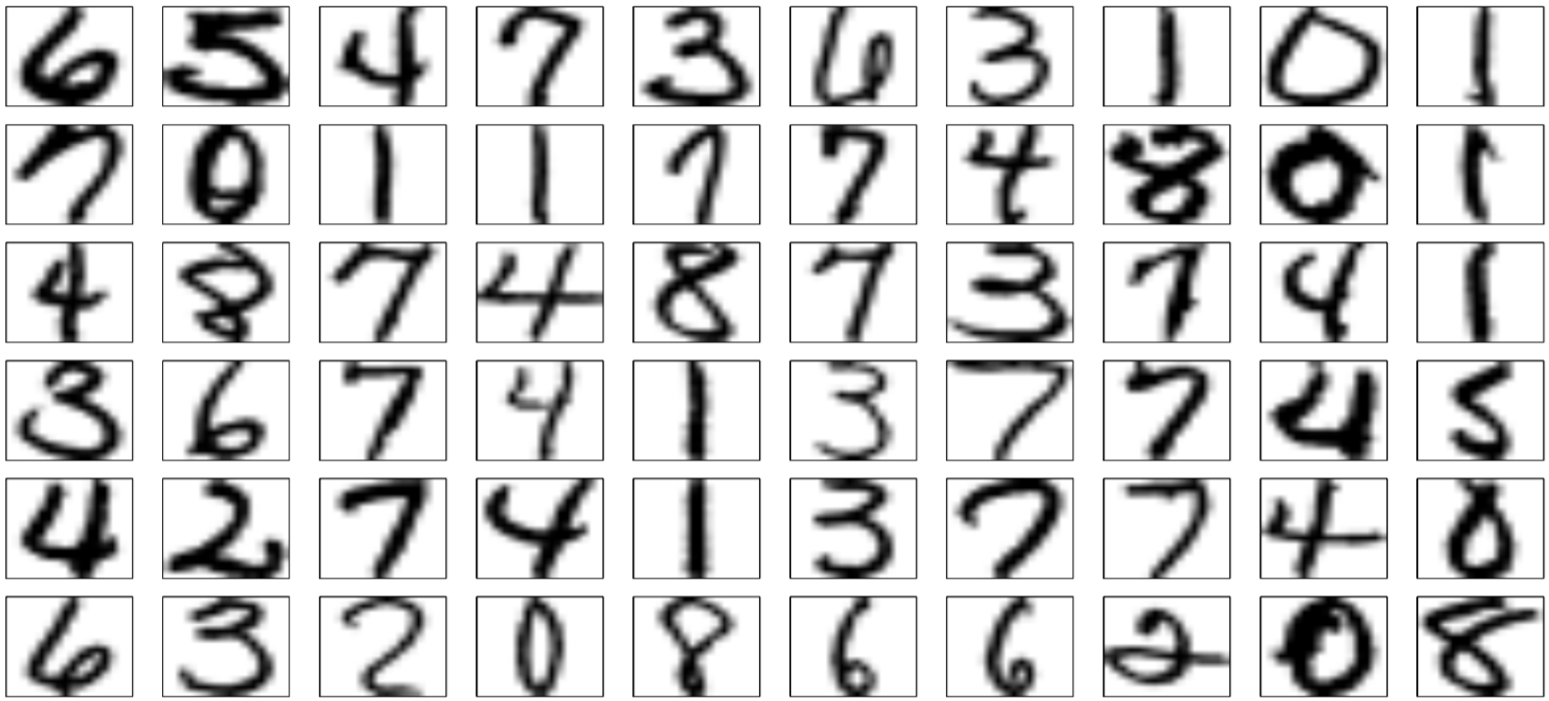
Features

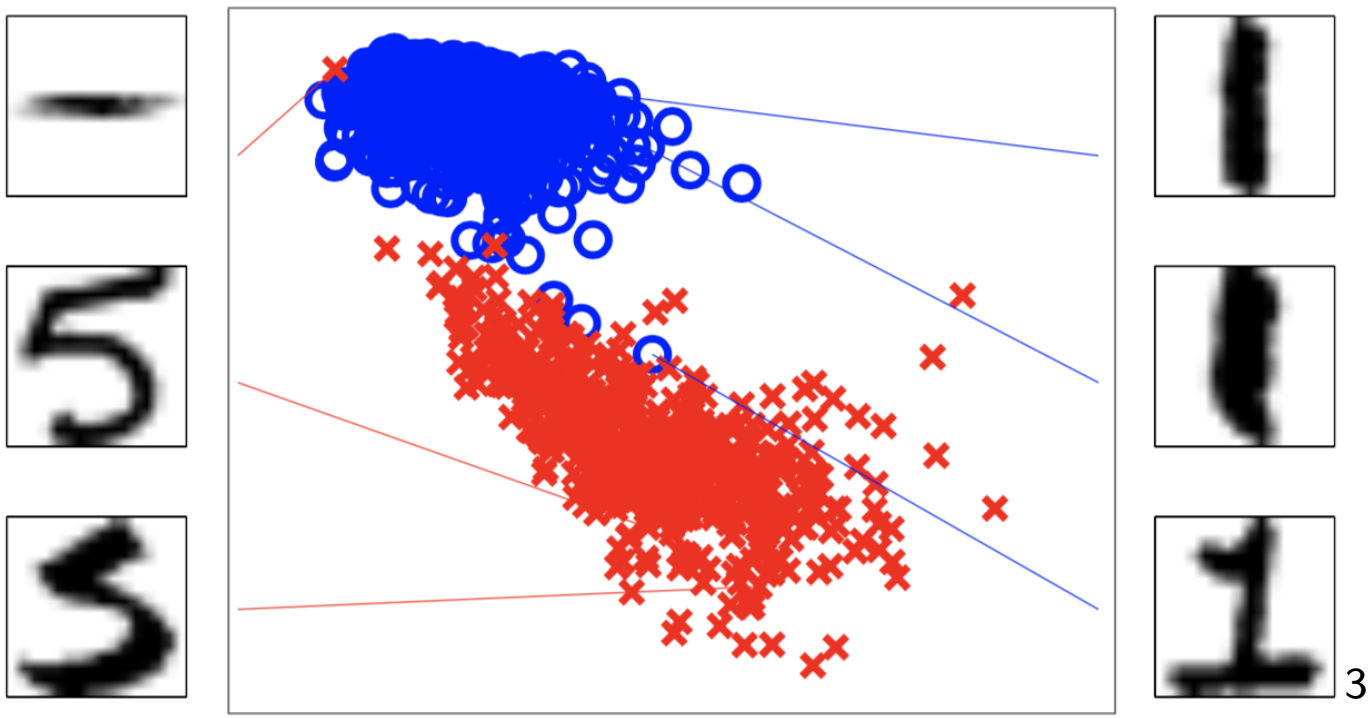
Remember that the target function we're trying to learn has the form $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ is typically a matrix of feature vectors and $\mathcal{Y}$ is a vector of corresponding labels (classes). Consider the problem of classifying images of hand-written digits:



What should the feature vector be?

Feature Engineering

Sometimes deriving descriptive features from raw data can improve the performance of machine learning algorithms.



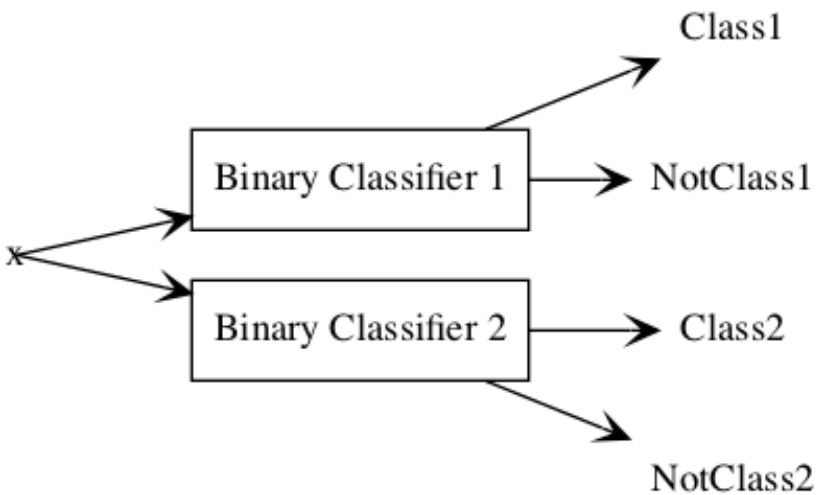
Here we project the 256 features of the digit images (more if you consider pixel intensity) into a 2-dimensional space: average intensity and symmetry.

[3] http://www.cs.rpi.edu/~magdon/courses/learn/slides.html

Multiclass Classification

We've only discussed binary classifers so far. How can we deal with a multiclass problem, e.g., 10 digits?

- Some classifiers can do multi-class classification (e.g., multinomial logistic regression).
- Binary classifiers can be combined in a chain to handle multiclass problems



This is a simple example of an ensemble, which we'll discuss in greater detail in the second half of the course.

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Closing Thoughts

- Most data sets are not linearly separable
  - We minimize some error, or loss function
- Learning algorithms learn in one of two modes:
  - Online learning algorithm – model is updated after seeing one training samples
  - Batch learning algorithm – model is updated after seeing all training samples
- We've now seen hyperparamters to tune the operation of learning algorithms
  - T or ϵ to bound the number of learning iterations
  - A learning rate, α or η, to modulate the step size of the model update performed in each iteration
- A multiclass classification problem can be solved by a chain of binary classifiers

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