Linear Separators

The Perceptron Learning Algorithm

Learning Problem Setup

Every machine learning problem contains the following elements:

- An input $\vec{x}$
  - (though, as we'll see later, $\vec{x}$ can be a list aof arbitrary feature values, not necessarily a vector)
- An unkown target function $f : \mathcal{X} \rightarrow \mathcal{Y}$
- A data set $\mathcal{D}$
- A learning model, which consists of
  - a hypothesis class $\mathcal{H}$, and
  - a learning algorithm.

A learning algorithm uses elements of $\mathcal{D}$ to estimate parameters of of a particular $h(\vec{x})$ from $\mathcal{H}$ which maps every $\vec{x}$ to an element of $\mathcal{Y}$.

Example: Credit Scoring

Let's create a credit score based on two variables: age and income (in thousands), which are real numbers.

- An input $\vec{x}$ is a vector in $\mathbb{R}^2$. For example, a 25 year-old person making \$60,000 would be represented by the vector $(24, 60)$.
- "Credit score" $= \sum_{i=1}^{d} w_i x_i$

In other words:

- Approve credit if $\sum_{i=1}^{d} w_i x_i >$ threshold
- Deny credit if $\sum_{i=1}^{d} w_i x_i <$ threshold

The weights w_i represent the importance of corresponding features of input instances.

Learning Model

Our credit score:

- Approve credit if $\sum_{i=1}^{d} w_i x_i >$ threshold
- Deny credit if $\sum_{i=1}^{d} w_i x_i <$ threshold

Can be turned into a learning model as

$$h(x) = sign((\sum_{i=1}^{d} w_i x_i) + w_0)$$

The “bias weight” w_0 corresponds to the threshold.

Our learning model is a discriminant – for an instance $\vec{x}$ it returns $+1$ or -1.

The Credit Data Set

Let's look at a particular data set, $\mathcal{D}$, stored in credit.csv

- Each data point represents a previous customer
- Since this is is supervised learning, every data point has an associated label: $+1$ for a customer off whom the bank made money, -1 for a customer off whom the bank lost money

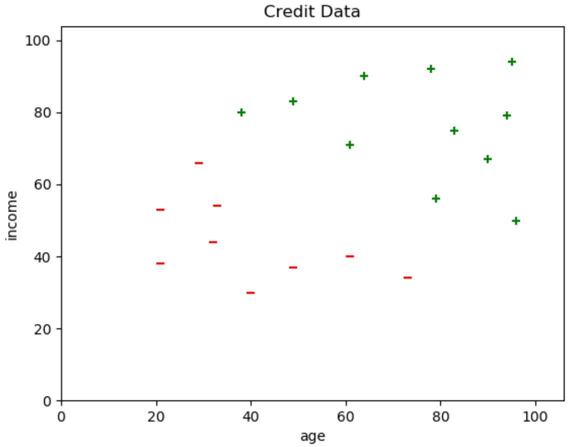
We can load the data set directly into a DataFrame:

```
In [34]: credit = pd.read_csv(credit.csv)

In [35]: credit
Out[35]:
    age  income  approve
0    64      90        1
1    78      92        1
2    38      80        1
3    29      66       -1
4    94      79        1
... additional rows elided
```

Credit Data Plot

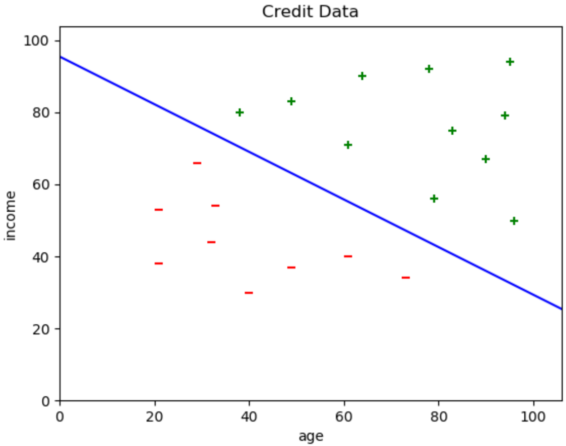
A scatter plot gives us intuition about the structure of the data.



Is there a line that separates the +s from the -s?

A Linear Separator

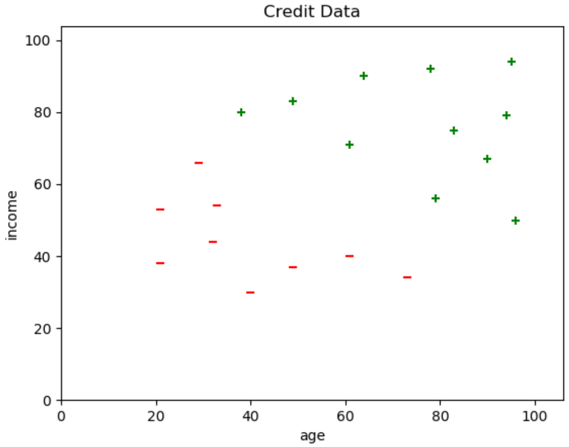
Here's a line that separates the data into classes.



Are there other lines? How many of these lines are there?

Version Spaces

A *version space* is the set of all h in $\mathcal{H}$ consistent with our training data. For our credit data, it's the set of all lines that separate the classes.



Now the question is, how can we find one of these lines

The Perceptron Hypothesis Class

The set of lines (more generally, hyperplanes) is a hypothesis class known as the *perceptron*.

$$\mathcal{H} = \{h(\vec{x}) = sign(\vec{w}^T \cdot \vec{x})\}$$

where

$$\vec{w} = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_d \end{bmatrix} \in \mathbb{R}^d \qquad \vec{x} = \begin{bmatrix} 1 \\ x_1 \\ \vdots \\ x_d \end{bmatrix} \in \{1\} \times \mathbb{R}^d$$

Notice that we enode the bias term as w_0 and we prepend all data points with 1 so that a simple dot product of the weight vector with a data point gives us the weighted sum we formalized earlier.

The Perceptron Learning Algorithm

A simple algorithm to find a linear separator is the perceptron learning algorithm:

Given a data set $\mathcal{D}$ with each $\vec{x_i}$ in $\mathcal{D}$ prepended with a 1, and labels $\vec{y}$

1. Initialize $\vec{w} = (w_0, w_1, ..., w_d)$ with zeros or random values.
2. As long as there is a $\vec{x_i}$ in $\mathcal{D}$ for which $sign(\vec{w}^T \cdot \vec{x}) \neq y_i$
    - Update $\vec{w}$ using the update rule:
        - $\vec{w}(t+1) = \vec{w}(t) + y_i\vec{x_i}$ (t is "time", or iteration number)

When the algorithm finishes, $\vec{w}$ is a line separating the two classes (assuming $\mathcal{D}$ is linearly separable).

Let's see this algorithm in action. . .

The General Form of Learning Algorithms

The Perceptron Learning Algorithm exemplifies the general form of learning algorithms

1. Initialize a model's parameters to some initial values.
2. Until some stopping criterion is reached (e.g., error within bounds)
    - Evaluate the model on some subset of the data $\mathcal{D}$
    - If error is present, update the model's parameters to reduce the error
        - The magnitude of the correction is often captured in a "learning rate" hyperparameter, often represented by η or α

When the algorithm is finished, you have a model, a particular $h \in \mathcal{H}$, that "fits" the training data.

Closing Thoughts

The perceptron model and the perceptron learning algorithm give us a simple, easy to grasp introduction to general issues in machine learning.

In the coming weeks we will develop a more sophisticated conceptual and algorithmic toolkit.

- For our PLA we chose a data set with a strong (and rare) property: perfect linear separability. What if we have data that are not linearly separable?
- How do we model error?
- How do we deal with noise?
- What if our data can't be represented as vectors?